Twelve coins problem

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1 Problem

Of twelve coins, one is counterfeit and weighs either more or less than all the others. The others weigh the same. With a balance scale, on which one side may be weighed against the other, you are to use only three weighings to determine the counterfeit [1].

2 Solution

Let M be the set of coins, $|M| = 12$. We have weighing function

$$w : M \to \{a, b\},\ a \neq b,\ a, b \in \mathbb{R}^+.$$

We have $|\{c \in M : w(c) = a\}| = 11$ and $|\{c \in M : w(c) = b\}| = 1$. We are asked to find $c_f \in M$ with $w(c_f) = b$ in three weighings.

For a subset $S \subseteq M$ we define

$$w(S) = \sum_{c \in S} w(c).$$

Let's partition M into 3 subsets S_0, S_1, S_2

$$\begin{array}{l} S_0 \cup S_1 \cup S_2 = M \\ \forall\ 0 \leq i < 3 : |S_i| = 4 \\ \forall\ 0 \leq i < j < 3 : S_i \cap S_j = \emptyset \end{array}$$

At this point we consume the first weighing:

1st weighing: compare $w(S_1)$ with $w(S_2)$

2.1 $w(S_1) = w(S_2)$

In this case $c_f \in S_0$. We partition S_0 into $S_0 = S_0^1 \cup S_0^3$ with $|S_0^1| = 1$ and $|S_0^3| = 3$. We also consider S_1^3, a subset of S_1 with $|S_1^3| = 3$. We consume the second weighing:

2nd weighing: compare $w(S_0^3)$ with $w(S_1^3)$

case 1: $w(S_0^3) = w(S_1^3)$**.** In this case $c_f \in S_0^1$ and we're done after just two weighings.

case 2: $w(S_0^3) > w(S_1^3)$**.** In this case S_0^3 has the counterfeit coin and $b > a$. We consume the third weighing: Let $S_0^3 = \{c_1, c_2, c_3\}$. We weigh c_1 against c_2.

3rd weighing: compare $w(c_1)$ with $w(c_2)$

If $w(c_1) = w(c_2)$ then $c_f = c_3$, if $w(c_1) > w(c_2)$ then $c_f = c_1$.

case 3: $w(S_0^3) < w(S_1^3)$**.** In this case S_0^3 has the counterfeit coin and $b < a$. Analog to previous case (replace heavy with light).

2.2 $w(S_1) > w(S_2)$

In this case the counterfeit coin is either in S_1 or in S_2.
We consider 4 subsets:

$S_0^3 \subset S_0, |S_0^3| = 3,$

A with three coins from S_1,

B with one coin from S_2,

C with remaining coin from S_1: $C = S_1 \setminus A$.

We consume the second weighing:

2nd weighing: compare $w(S_0^3 \cup C)$ with $w(A \cup B)$

case 1: $w(S_0^3 \cup C) = w(A \cup B)$**.** In this case $c_f \in S_2 \setminus B$ and because $w(S_1) > w(S_2)$ we know that $b < a$. Let $\{c_1, c_2, c_3\} = S_2 \setminus B$ and we consume third weighing:

3rd weighing: compare $w(c_1)$ with $w(c_2)$

If $w(c_1) = w(c_2)$ then $c_f = c_3$, if $w(c_1) > w(c_2)$ then $c_f = c_2$.

case 2: $w(S_0^3 \cup C) < w(A \cup B)$**.** Assume $c_f \in C \subset S_1$. That would mean that $b > a$ because $w(S_1) > w(S_2)$ but that contradicts with $w(S_0^3 \cup C) = 3a+b < w(A \cup B) = 4a$. Assume $c_f \in B \subset S_2$. That would mean that $b < a$ because $w(S_1) > w(S_2)$ but that contradicts also with $w(S_0^3 \cup C) = 4a < w(A \cup B) = 3a + b$. The only possibility remaining is $c_f \in A$. We use the ***third weighing*** analog to the previous case to find the counterfeit coin in a three-coin set using the fact that $b > a$.

case 3: $w(S_0^3 \cup C) > w(A \cup B)$**.** In this case the counterfeit coin can be either in B or in C. It cannot be in A according to a reasoning analog to previous case that leads to a contradiction. Both B and C only have one coin each so compare the coin in B with any good coin to find the counterfeit coin in this ***third weighing***.

This covers all the cases and we're done.

References

[1] Ethan Canin. *The Palace Thief Stories*, chapter Batorsag and Szerelem, page 87. Random House New York, 1994.